Numerical Simulation of the Unsteady Wake Behind an Airfoil

D. T. Mook,* S. Roy,† G. Choksi,† and B. Dong†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia

The unsteady wake behind an airfoil is simulated numerically by a system of discrete vortex cores, also called point vortices. In common with previously developed procedures, at each time step a core is added to the wake at the trailing edge, and the cores already in the wake are convected at the local particle velocity. The innovation of the present method is that, as the cores begin to separate, more cores are added to the system and the circulations around the individual cores are reduced according to a linear interpolation routine. The spacing between cores is maintained approximately while the total circulation around the wake and airfoil is maintained exactly. In several examples, one finds qualitative agreement between computed wake shapes and flow visualization. The method shows promise as a means of simulating unsteady, closely coupled aerodynamic interference.

Introduction

THE problem of developing a numerical model for general unsteady, incompressible, two-dimensional flows over airfoils is one of long-standing interest. In order to predict the unsteady downwash on an airfoil accurately, one must have an accurate simulation of the near-field wake. Moreover, to model closely coupled aerodynamic interference, especially the situation in which one airfoil is operating in or near the wake of another, one must have an accurate simulation of the medium-field wake. For these reasons, among others, an accurate numerical simulation of the wake is of paramount importance in general unsteady aerodynamics.

In a recent paper, Kim and Mook[1] described a technique that uses a piecewise linear, continuous distribution of vorticity to represent the surface of the wing, and a system of discrete vortex cores (sometimes called point vortices) to represent the wake. They gave a brief account of earlier attempts to model this flowfield. (The reader is referred to this paper for the references.) The use of continuous vorticity panels to represent the airfoil in unsteady flows was, apparently, unique; however, all successful previous models used discrete cores to represent the wake. The use of discrete cores to represent a continuous vortex sheet (in this case, the wake) apparently dates back to Rosenhead.[2] Although the practice is widespread, it has been questioned a number of times; see, e.g., the recent article by Krasny,[3] which reviews the literature and discusses the formation of a singularity. It appears that Krasny considered a fixed number of cores in his discretized representation of a vortex sheet. Here we continuously add cores and so far have not encountered any problem, even after computing over considerable lengths of time. The discrete vortex cores are convected with the local particle velocity. For an inviscid fluid, this is done to satisfy the requirement that pressure be single-valued in the wake. The wake, therefore, is treated as a region that contains vorticity, but one in which viscous effects are negligible. Apparently, the extreme deformations (sometimes called rolling up) experienced by the wake preclude the use of some sort of continuous representation. At least Kim and Mook (and presumably the others) were not successful in their attempts to develop a continuous representation.

The use of discrete cores to represent the wake is not without problems. Because the velocity field induced by a core is computed according to potential-flow theory, it contains a strong singularity at the core (the speed becomes infinite as $1/r$ when r, the distance from the core to the point where the speed is being calculated, approaches zero). This eventually leads to the physically unrealistic situation in which two discrete cores that had been close at previous time steps are suddenly convected far apart. This problem can be reduced, or delayed, but never completely eliminated in a realistic fashion, by the introduction of a so-called cutoff length (if the field point where the velocity is being calculated is a distance less than the cutoff length from the core, the velocity is set equal to zero) and/or by the combination of cores when they are first convected very close.

In the present paper, we describe a recent innovation that overcomes these problems and leads to an apparently accurate simulation of the wake. At the least, there is strong qualitative agreement between predicted results and flow visualization. In common with many earlier practices, we use a cutoff length. In contrast with some earlier practices, we do not combine cores. Instead we do just the opposite: we divide the cores into smaller ones (i.e., ones having less circulation) as the wake forms. At the same time, the sum of the circulations around all the cores in the wake remains constant. This is done repeatedly, so that after one hundred or so time steps, after one hundred or so cores have been shed from the trailing edge into the wake, there are typically one thousand or more cores representing the wake. The details of this relatively simple procedure and several numerical examples are presented below.

Brief Outline of the Numerical Method

Here we briefly outline the procedure developed by Kim and Mook, giving particular attention to the details at the trailing edge and the manner in which vorticity is shed into the wake. The airfoil is wrapped in a vortex sheet; consequently, it is possible to use the tangential component of the relative velocity at the surface as the unknown. We call this tangential component the surface velocity. The actual surface of the airfoil is approximated by a series of connected, short, straight segments. The actual surface velocity is approximated by a continuous function that varies linearly across each straight segment. The values of the surface velocity at the junctions of the straight segments are the unknowns in the numerical problem. These values of surface velocity are obtained by imposing the no-penetration boundary condition in some optimal sense. An appealing feature of this approach is that a primitive variable, not a potential one, serves as the unknown.

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*Professor, Engineering Science and Mechanics. Member AIAA.
†Graduate Research Assistant, Engineering Science and Mechanics.

The discussion of the trailing edge given in Ref. 1 is expanded. (Several references to earlier treatments are given in Ref. 1 and not repeated here.) The basic requirement is that the pressures along the upper-surface and lower-surface streamlines approach the same value at the trailing edge.

Bernoulli's equation may be written in the form

$$Cp=\frac{P-P_\infty}{\frac{1}{2}\rho V_\infty^2}=1-V^2-2\frac{\partial\phi}{\partial t} \tag{1}$$

where P is the local pressure; P_∞ is the pressure in the far field; V_∞ is the speed of the uniform stream in the far field or the characteristic forward speed of the reference point in the airfoil, depending on how the problem is posed; ρ is the density (a constant); V is the local dimensionless speed of the fluid; and ϕ is the dimensionless velocity potential. Using Eq. (1), one finds that the basic requirement is satisfied if

$$2\left(\frac{\partial\phi_u}{\partial t}-\frac{\partial\phi_\ell}{\partial t}\right)=2\frac{\mathrm{d}\Gamma_\beta}{\mathrm{d}t}=V_\ell^2-V_u^2 \tag{2}$$

where the subscripts u and ℓ denote the upper and lower surfaces, and Γ_β refers to the circulation around the airfoil. The circulation is obtained by integrating around the airfoil in a clockwise direction. Equation (2) indicates that for unsteady flow ($\mathrm{d}\Gamma_\beta/\mathrm{d}t\neq 0$), the speeds along the upper and lower surfaces must be different. This is a manifestation of a vortex sheet at the trailing edge.

In the numerical model, the speed on the surface is approximated by a piecewise linear, continuous function. As a matter of numerical convenience, this function is chosen such that it is zero on both the upper and lower surfaces at the trailing edge. This is clearly inconsistent with the conclusions stated in the discussion above, so a discrete vortex core (sometimes called a point vortex) is added to the flowfield at the trailing edge. (The details are given in Ref. 1.) This core may be viewed as a further approximation; it generates a velocity field that approximates the one generated by the nonzero surface velocity over the last very short distances on the upper and/or lower surface. The circulations around the trailing-edge cores change sign according to whether Γ_β is increasing or decreasing. To simulate the requirement of equal pressures on the upper-surface and lower-surface streamlines at the trailing edge, we consider this core to be a free vortex; thus, the core present at the trailing edge at any instant is shed and convected away at the local velocity. As one core is shed, another is simultaneously formed. It in turn is shed, and another is formed. The circulation around each core is predicted when it first appears and subsequently remains constant until the core is split. Splitting is discussed later. The cores are the numerical model of the wake. The use of cores to represent the wake is not new, nor is it without problems. This point is discussed later as well.

The no-penetration boundary condition is imposed at the midpoints between the junctions of the linear segments. In addition to the no-penetration condition, the total circulation around both the airfoil and its wake remains zero. At this point, the number of equations is always one more than the number of unknowns; as a result, we obtain the surface velocity and the circulation around the core at the trailing edge by a least-square minimization of the error in the no-penetration condition subject to the equality constraint that the circulation around the airfoil and its wake remains zero. The details are given in Ref. 1.

There naturally arises the question of how well does the method work. This is difficult to answer because of the lack of exact solutions. In Ref. 1, several solutions obtained by the present method were found to be in close agreement with solutions obtained by others, and one solution appeared to be in close agreement with an experimental result. Splitting the cores in the wake does not affect the wake-generating airfoil significantly. (The benefits of splitting are found in or very near the wake downstream.) Consequently, the present method yields virtually the same results presented in Ref. 1. Some results that were not given in Ref. 1 are discussed next.

First, we consider the numerical simulation for the flow around an impulsively started airfoil. At the instant following an impulsive start, there is one core located at the trailing edge. The computed streamline pattern and the computed surface-velocity distributions are shown in Fig. 1 for 36, 72, and 108 elements. The distance along the surface of the airfoil

Table 1 Circulation around the core at the trailing edge at the instant of an impulsive start for different numbers of elements to represent the surface of an airfoil

Number of elements	Γ around the trailing edge core
36	−0.1269
72	−0.0637
108	−0.0425

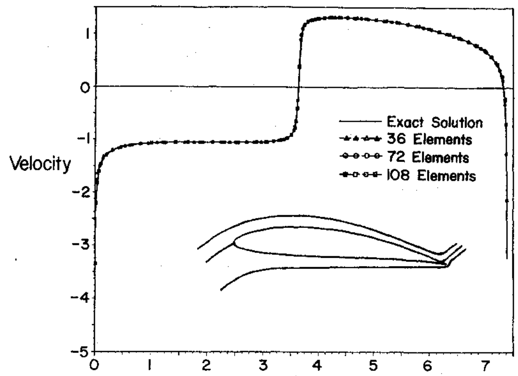

Fig. 1 Velocity on the surface of a Karman–Trefftz airfoil at the instant following an impulsive start as a function of distance along the surface from the point on the lower surface at the trailing edge. The insert shows the corresponding streamline pattern.

Table 2 Circulation around the wake Γ_w and its derivative for different numbers of elements and time-step sizes when the airfoil has traveled 0.02 chord after an impulsive start. The time step is given by $\Delta t = 0.075/N$

	36 elements		72 elements		108 elements	
N	Γ_w	$\mathrm{d}\Gamma_w/\mathrm{d}t$	Γ_w	$\mathrm{d}\Gamma_w/\mathrm{d}t$	Γ_w	$\mathrm{d}\Gamma_w/\mathrm{d}t$
1	0.3040	1.484	0.2654	1.479	0.2392	1.522
2	0.3101	1.529	0.2799	1.442	0.2646	1.420
4	0.3132	1.488	0.2866	1.438	0.2758	1.404
8	0.3143	1.516	0.2897	1.439	0.2810	1.405
16	0.3150	1.515	0.2912	1.440	0.2834	1.405
32	0.3153	1.511	0.2919	1.439	0.2846	1.407
64	0.3155	1.512	0.2923	1.439	0.2852	1.406

[a]Circulation around the wake Γ_w and its derivative for different numbers of elements and time-step sizes when the airfoil has traveled 0.02 chord after an impulsive start.

from the point on the lower surface at the trailing edge is labeled S. The point on the upper surface at the trailing edge corresponds to $S = 7.45$ approximately. The first and last points where the speed is computed in the numerical simulation are very close to, but not at, the trailing edge. The sign of the surface velocity agrees with the sign of the corresponding vorticity in the vortex sheet; clockwise rotation is positive.

In accord with the discussion of the trailing edge given previously, the core at the trailing edge is considered free (i.e., convecting at the local fluid-particle velocity). But, because the core is located at the trailing edge and the total circulation around both airfoil and wake (i.e., core) is zero, the flowfield given by the present method approximates the exact solution for flow around the airfoil at zero circulation. This exact solution is also given in Fig. 1. The present method predicts the forward flow on the upper surface near the trailing edge and the location of the rear stagnation point accurately.

The predicted streamline pattern in Fig. 1 is in very good agreement with the pattern in a Hele-Shaw flow past an airfoil at incidence. The Hele-Shaw flow cannot show circulation; consequently, infinite velocities are represented at the trailing edge.[4] Thus, the Hele-Shaw flow corresponds to the instant the motion begins, the same as the numerical simulation represented in Fig. 1.

At this instant, there is only one core in the flowfield, and it is located at the trailing edge. As one would expect for the present method of approximating the potential flow, the circulation around the core decreases and the surface speeds at the points nearest the trailing edge increase correspondingly as the number of elements increases. The results are shown in Table 1.

When the numerical simulation proceeds, the core at the trailing edge is shed, displaced by $v\Delta t$ where v is the velocity at the trailing edge and Δt is the time step, and a new core is added to the flowfield at the trailing edge. Kim and Mook experimented with other algorithms to predict the trajectories of the cores and found this simple Euler algorithm to be adequate. Then the optimization is repeated; the velocity field induced by the wake is now the sum of the contributions from both cores. This can be repeated for any desired number of time steps. In Table 2, some results are presented for different time-step sizes and numbers of elements. For all of these results, the airfoil has advanced 0.02 chords after the impulsive start.

After the impulsive start, the rearward stagnation point on the upper surface rapidly moves aft and disappears from the computed results. When this happens, the interpretation is that the flow along the upper-surface streamline sees a stagnation point at the trailing edge ($V_u = 0$), while the flow along the lower surface streamline leaves the airfoil tangent to the lower surface at finite velocity ($0 < V_\ell < \infty$). The results shown in Fig. 2a seem to substantiate this interpretation. If the airfoil stops, then the situation changes. The circulation around the airfoil begins to decrease and correspondingly the circulations around the trailing-edge cores change sign. If only one stagnation point is computed, then the interpretation is that the flow along the lower-surface streamline sees a stagnation point ($V_\ell = 0$), while the flow along the upper-surface streamline leaves the airfoil tangent to the upper surface at finite velocity ($0 < V_u < \infty$). The results shown in Fig. 2b seem to substantiate this interpretation. The conditions given in Fig. 2 evolve naturally as the computation proceeds.

In Fig. 3, the predicted pressure distributions on the airfoil surface are given at various times after the impulsive start. All of the results were computed with 36 elements. Initially, there is a relatively high downward loading near the trailing edge as a result of the stagnation point on the upper surface. As the airfoil moves forward, the high loading decreases and the pressure distribution starts to resemble that for steady flow; the results are shown in Fig. 3b after the airfoil has moved 1.9 chords. Steep pressure gradients near the trailing edge appear for nearly all stages of the flow past this airfoil. The numerical results for the steady flow in Fig. 3c are in virtually perfect agreement with the exact solution when the so-called Kutta condition is imposed (i.e., when both V_u and V_ℓ are zero). More results and explanations are given by Dong.[5]

The evolution of these pressure distributions is in close qualitative agreement with some recent results of Teng,[6] who computed the unsteady pressure distribution on a symmetric von Mises airfoil after an impulsive change in angle of attack. Teng used panels of constant-strength sources and a constant-strength vortex sheet for surface singularities. His wake consisted of a system of discrete vortex cores as in the present model, and a single vortex panel adjoined to the trailing edge.

Root of the Trouble with the Earlier Numerical Simulations of the Wake and an Earlier Remedy

In Fig. 4, the locations of the cores in the computed wake behind an airfoil that is experiencing a harmonic plunging mo-

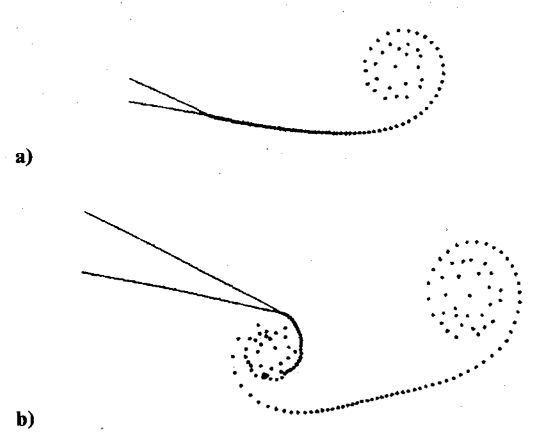

Fig. 2 Locations of cores in the wake of airfoil: a) shortly after an impulsive start, and b) shortly after an impulsive start followed by an impulsive stop.

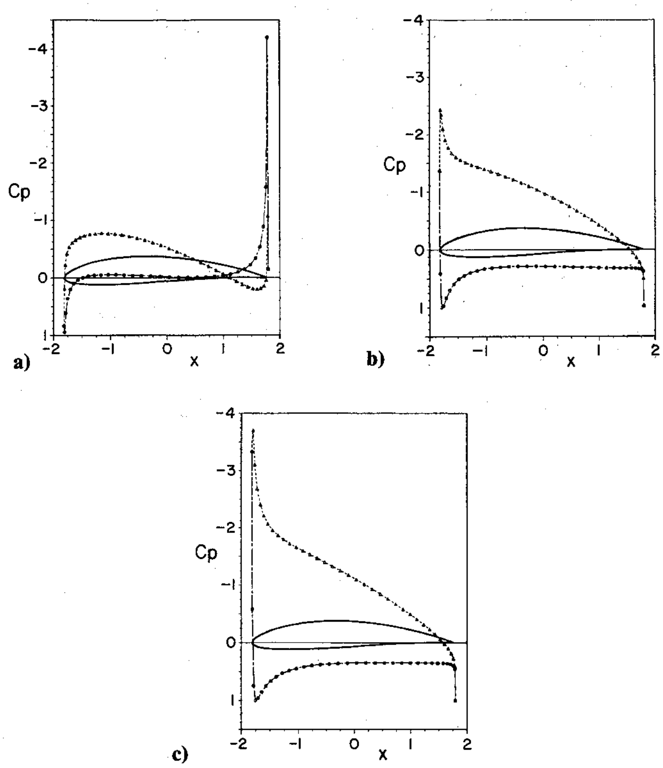

Fig. 3 Predicted pressure distribution on the surface of a Karman–Trefftz airfoil following an impulsive start: a) after the airfoil has traveled 0.0055 chord; b) after 1.889 chords; and c) steady state.

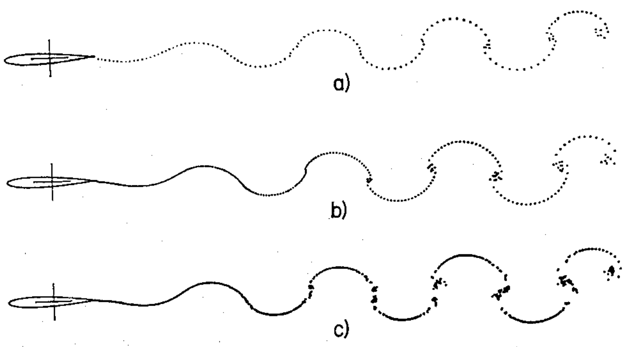

Fig. 4 Wakes behind a plunging airfoil computed in the conventional way: a) 128 time steps; b) 256 time steps; and c) 512 time steps.

tion are shown. The amplitude and frequency of the motion of the airfoil are the same in the three parts. The three parts of the figure cover precisely the same time period, but the time-step sizes are different: 128, 256, and 512 steps are used in Figs. 4a–4c, respectively. These results were obtained in what might be termed the conventional way; no special procedure to redistribute, combine, or split cores was used. Because a core is added at every time step, the number of cores in Fig. 4c is twice that in Fig. 4b, which is twice that in Fig. 4a. The cores were not connected so that one can clearly see where they are located. In Fig. 4c, the cores are so densely packed near the trailing edge that the wake appers to be a continuous line.

The root of the problem caused by the conventional use of vortex cores is evident in these pictures. There are regions where the cores are separating, leaving big gaps compared with the situation near the trailing edge; and there are regions where the cores are beginning to reassemble or collect. Eventually, this unequal spacing leads to serious computational problems. The results in all three parts show the same trends (in fact, they suggest a robust trend to converge as the step size decreases); consequently, decreasing the time step and thereby adding more cores (or combining the cores in the regions where they collect) will not eliminate the problem.

The elimination of the unequal spacing of cores in the wake has been the subject of several efforts. The earlier efforts have involved combining cores in the regions where they tend to collect and redistributing the cores at even spatial intervals. Sarpkaya and Schoaff[7] described a redistribution scheme and provided a review of earlier attempts along similar lines. They used the same Euler scheme used here to calculate the displacements of the cores in the wake. At each time interval, after performing the convection and before proceeding to the new time (i.e., calculating the downwash of the wake on the cylinder), they repositioned the cores. This repositioning process involved the following: 1) separating the cores by equal distances along nearly the same line they occupied before relocation, 2) distributing the circulations around the newly placed cores in a way that closely approximates the spatial distribution before relocation, and 3) maintaining the total circulation of the wake.

In the present paper, we present an alternative to this procedure. Here we add cores as the distances between them increase.

Present Scheme for the Maintenance of Nearly Equal Spacing of Cores and Some Numerical Examples

The tendency for the distance between the cores in the numerical simulation of the wake to increase suggests that in the actual flow the region containing the vorticity tends to become elongated. This conclusion seems to be substantiated by the recent experiments of Poling[8] among others. To imitate this behavior, we split the cores into parts in such a way that the total circulation remains constant. Specifially, as soon as two cores that were shed successively separate by more than a prescribed amount, a new core is placed at the midpoint of the

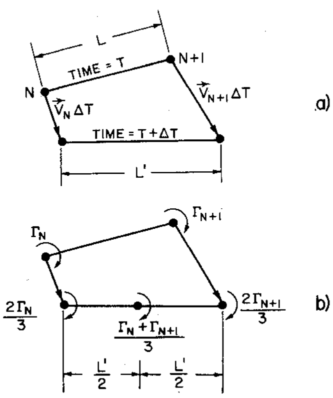

Fig. 5 a) The conventional convection scheme and b) the present scheme.

Fig. 6 Wake behind a plunging airfoil computed by the present scheme. These results are to be compared with Fig. 4b.

line segment connecting them, the new core is given circulation equal to one-third the sum of the circulations around the two original cores, and the circulations around the two original cores are reduced by a factor of two-thirds. In this way, the sum of the circulations around all the cores in the wake remains constant. After the addition of a core, the cores are renumbered and the procedure is repeated until the distances between all successive cores are considered. The entire procedure is executed at each time step.

In Fig. 5, the procedure is illustrated. In Fig. 5a, two typical cores are shown at two times; their displacements during the time interval ΔT are $V_N \Delta T$ and $V_{N+1} \Delta T$, where the subscripts refer to the core numbers. At time T, the distance between the cores L was less than a user-chosen critical length; at time $T+\Delta T$, the distance between the cores L' is greater than the critical length. Consequently, at time $T+\Delta T$, the present procedure replaces the two cores with three cores as shown in Fig. 5b. The new core lies at the midpoint of the line segment connecting the two orginal cores. The circulations around the three cores are indicated in the figure; the sum of the circulations around the three cores is the same as the sum around the original two cores. Consequently, the spacing of the cores is maintained approximately while the total circulation around the entire wake and airfoil is maintained exactly. Pullin and Jacobs[9] apparently used a similar scheme in connection with the calculation of the evolution of a vortex sheet. Pullin and Jacobs also indicated that they combined cores when they came close together. We experimented with combining cores, but found that the results appear to be less accurate. All of the results presented here are based on addition only. For the results presented here, the time step was chosen to be numerically equal to the length of the first element. The cores were split as soon as the distance between them increased to 1.5 of this length; thus, all of the distances between cores lie in the range of 0.75–1.5 of this length.

In Fig. 6, a wake computed by the core-addition scheme is shown. These results correspond to those given in Fig. 4b; the step sizes are the same and both sets of results cover precisely four cycles of the motion. The similarity is very evident, but so are significant differences. The vortex sheet in Fig. 6 shows formations of wavelets, which do not appear in Fig. 4. The cores tend to collect in the same general areas in the two figures, but in Fig. 6 several small regions appear within each large group. In fact, the results in Fig. 6 compare better with those in Fig. 4c than with those in Fig. 4b. Several runs with different step sizes and prescribed distances for which cores

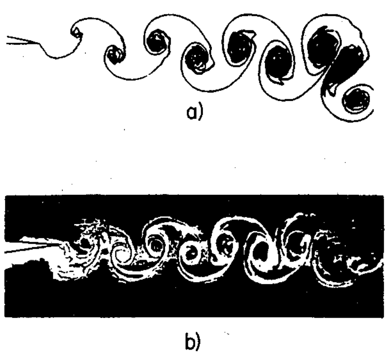

Fig. 7 Comparison of a) computed wake shape with b) flow visualization (Bratt[10]).

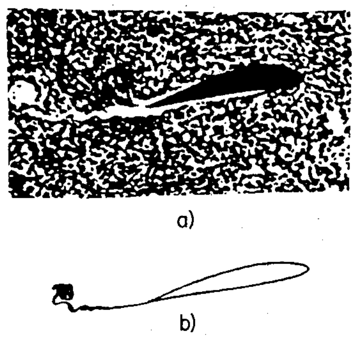

Fig. 8 Comparison of a) flow visualization[11] with b) computed wake shape after an impulsive start.

are added show that the results exhibit a strong trend to converge. Next, we present some additional results that further illustrate the present method.

In Fig. 7, the computed results are compared with flow visualization. In Fig. 7a, the computed wake for an airfoil pitching around its quarterchord point is shown. (Here the frequency of the motion is much higher than that used to compute the plunging motion illustrated in Fig. 6. The wakes for plunging and pitching at the same frequency are quite similar.) The large, dark regions, or clouds, are actually groups of very large numbers of cores. The intensity of the vorticity in these regions is rather low. Although there is no viscosity in the present model of the wake, the results seem to imitate diffusion of vorticity. In Fig. 7b, the experimental results of Bratt[10] are shown. Bratt introduced smoke upstream from the airfoil. The smoke appears white in the figure. The trailing edge of the airfoil appears at the left side. There is good qualitative agreement; the vertical and horizontal spacing of the core-like regions are nearly equal in the two figures.

In Fig. 8, the computed wake behind an airfoil impulsively started in translation at an angle of attack is compared with the results of a flow-visualization experiment in a well-known classroom film loop.[11] The presence of wavelets and a large core-like region are evident in both.

The problem of vortex-airfoil (or blade) interaction is one of long-standing interest. The recent paper by Poling et al.[12] gives a review and contains references to earlier work. In this paper, a group of vortex cores is released upstream from a stationary airfoil. As the group convects downstream, the relative positions of the cores change. Some cores pass over while others pass under the airfoil. The number of cores remains constant; core splitting was not used. The paper also gives some experimental results for the velocity field behind a pitching airfoil. The present method of simulating the wake is well suited for this problem. In Figs. 9 and 10, the computed wakes and the velocity fields are shown for two airfoils executing simple harmonic oscillations in pitch around the quarter chord. There is a uniform horizontal stream from left to right. In Fig. 9, the airfoil is alone and the wake is beginning to show the familiar pattern. In Fig. 10, there is a second airfoil in the flow. The second airfoil remains stationary, but both airfoils generate wakes. In both figures, the pitching airfoil is just completing the first cycle of the motion. Comparing the two figures, one can see the effect of the second airfoil on the wake of the first. The wake is divided; some cores pass over the top of the second airfoil while others pass along the bottom. The results in both cases cover precisely the same time period, but there are many more cores in the wake of Fig. 10 than in the wake of Fig. 9. Because the wake is simulated by many relatively weak, densely packed cores, the computed velocity field is quite smooth.

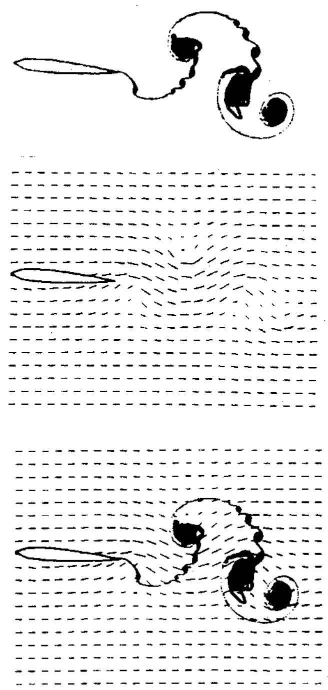

Fig. 9 Computed flowfield for an airfoil pitching in simple harmonic motion around the quarter-chord axis: a) the positions of the vortex cores in the wake; b) the corresponding velocity field; and c) wake and velocity field superposed.

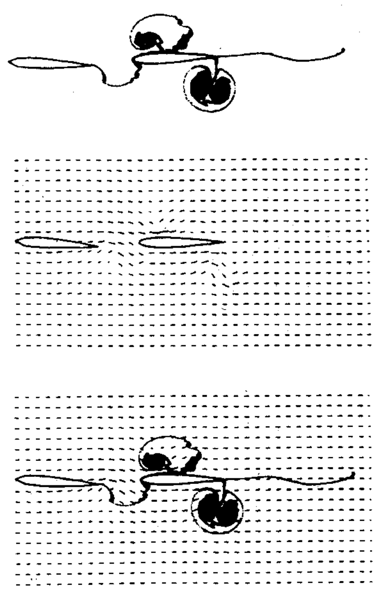

Fig. 10 Computed flowfield for one airfoil pitching around the quarter-chord axis and a second stationary airfoil in its wake. The motion of the pitching airfoil and the time period represented are identical to those in Fig. 9.

Concluding Remarks

The present scheme of numerically simulating the wake behind an airfoil uses a system of discrete vortex cores (sometimes called point vortices) to represent a continuous vortex sheet. The wake is regarded as a region that contains vorticity, but one where viscous effects are ignorable. Such an approach is not new. The innovation of the present scheme is to add cores according to linear interpolation as they begin to convect apart, and not to combine cores as they are convected close together. Several numerical examples show the computed wake shapes to be in close qualitative agreement with flow visualization. The present scheme shows promise as a means of calculating unsteady closely coupled aerodynamic interference.

Acknowledgment

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